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BAIRE NUMBER OF THE SPACES OF UNIFORM ULTRAFILTERS*

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ABSTRACT

The Baire number is defined for a topological space without isolated points as the minimal size of the family of nowhere dense sets covering the space in question. We prove that in the case of $U(\kappa)$, the space of uniform ultrafilters over uncountable κ, the Baire number equals either ω_1 or ω_2, depending on the cofinality of κ. The results are connected to the collapsing of cardinals when using the quotient algebra $\mathcal{P}(\kappa)$ mod$[\kappa]^{<\kappa}$ as the notion of forcing.

The aim of the present paper is to evaluate the Baire number of $U(\kappa)$, the space of uniform ultrafilters on the discrete set of cardinality κ. Recall that for a topological space X without isolated points, its Baire number $n(X)$ is defined as the smallest size of the family of nowhere dense sets covering X. Our attention will be focused on uncountable κ, because $n(U(\omega))$ was fully discussed in [BPS] as follows: Denote by $\mathfrak{h} = \min\{\tau$: the Boolean algebra $\mathcal{P}(\omega)/\text{fin}$ is not $(\tau, \cdot, 2)$-distributive$\}$. Then $\mathfrak{h}$ is a regular cardinal, $\omega_1 \le \mathfrak{h} \le \min(cf(\mathfrak{c}), \mathfrak{b}, \mathfrak{s})$, and for $n(U(\omega))$ one has: (a) $\omega_2 \le n(U(\omega))$, (b) if $\mathfrak{h} < \mathfrak{c}$, then $\mathfrak{h} \le n(U(\omega)) \le \mathfrak{h}^+$, (c) if

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$\mathfrak{h} = \mathfrak{c}$, then $\mathfrak{c} \leq n(U(\omega)) \leq 2^{\mathfrak{c}}$, and all possibilities not violating (a), (b), and (c) may happen, depending on the actual set theory. See also [Do].

Here we show that for κ with uncountable cofinality the Baire number is not influenced by additional axioms of set theory (in fact, they are all equal to ω_1) and for uncountable κ with countable cofinality it appears very likely that the value is always ω_2.

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The notation used throughout the paper is the standard one. Small Greek letters $\alpha, \xi, \eta, \ldots$ denote ordinals, κ and λ always stand for infinite cardinals. $[\kappa]^{<\lambda}$ denotes the set $\{M \subseteq \kappa\colon |M| < \lambda\}$, similarly for $[\kappa]^{\leq\lambda}$ and $[\kappa]^{\lambda}$. A **uniformly almost disjoint family** (abbr. UAD) is a family $\mathcal{B} \subseteq [\kappa]^{\kappa}$ such that for any two distinct $B_0, B_1 \in \mathcal{B}$ one has $|B_0 \cap B_1| < \kappa$. For $M \subseteq \kappa$, $\widehat{M}$ denotes the set $\mathrm{cl}_{\beta\kappa} M \cap U(\kappa)$, where $\beta\kappa$ is the Čech–Stone compactification of κ.

Let us begin with a trivial observation showing the lower bound for the Baire numbers in question.

OBSERVATION: *For each infinite κ, $n(U(\kappa)) \geq \omega_1$. If moreover $\mathrm{cf}(\kappa) = \omega$, then $n(U(\kappa)) \geq \omega_2$.*

Indeed, since $U(\kappa)$ is compact, the first half is an immediate consequence of the Baire Category Theorem. If $\mathrm{cf}(\kappa) = \omega$, then the intersection of every countable decreasing sequence of a non-empty clopen subsets of $U(\kappa)$ has non-empty interior. This fact together with an easy transfinite induction is enough to verify that no family of size ω_1 of nowhere dense subsets of $U(\kappa)$ covers $U(\kappa)$.

We shall frequently use the forthcoming lemma that is a mild strengthening of a known statement due to W. Kulpa and A. Szymański [KS].

LEMMA 1: *Let X be a space without isolated points, and let τ be an infinite cardinal number. If there is a family $\{\mathcal{V}_\alpha\colon \alpha < \tau\}$ such that*

(a) *for each $\alpha < \tau, \mathcal{V}_\alpha$ is a pairwise disjoint family of non-void open subsets of X,*
(b) *for each $\alpha < \tau, |\mathcal{V}_\alpha| > \tau$,*
(c) *for each nonempty open $U \subseteq X$ there is some $\alpha < \tau$ such that for all $V \in \mathcal{V}_\alpha$, $U \cap V \neq \emptyset$,*

then $n(X) \leq \tau^+$.

Proof: For $\alpha < \tau$, enumerate $\mathcal{V}_\alpha$ as $\mathcal{V}_\alpha = \{V(\alpha,\xi)\colon \xi < |\mathcal{V}_\alpha|\}$ and define for $\gamma < \tau^+$

$$W_\gamma = \bigcup_{\alpha<\tau} \bigcup_{\gamma\leq\xi<\tau^+} V(\alpha,\xi).$$

This is clearly possible by (b). By (a), every set W_γ is a union of open sets, hence open. If $U \subseteq X$ is open and non-empty, then by (c) it meets all $V(\alpha,\xi)$ for some $\alpha < \tau$. Therefore $U \cap \bigcup_{\gamma\leq\xi<\tau^+} V(\alpha,\xi) \neq \emptyset$ and consequently $U \cap W_\gamma \neq \emptyset$ for every $\gamma < \tau^+$. So each set W_γ is dense. If $x \in X$ and $\alpha < \tau$, then by (a), there is at most one $\xi(\alpha) < \tau^+$ with $x \in V(\alpha,\xi(\alpha))$. Let $\gamma < \tau^+$ be such that $\gamma > \xi(\alpha)$ for all those $\alpha < \tau$ for which $\xi(\alpha)$ is defined. For this γ, we have that $x \notin W_\gamma$, which implies that $\bigcap_{\gamma<\tau^+} W_\gamma = \emptyset$. Passing to complements $X \setminus W_\gamma (\gamma < \tau^+)$, we obtain the family of nowhere dense sets covering X. ∎

Our next lemma is easy, too. Let us remind ourselves that a mapping $f\colon X \longrightarrow Y$ between two topological spaces is called **semi-open**, if for every non-empty open $U \subseteq X$ the interior of the image $f[U]$ is non-empty.

LEMMA 2: *Let X, Y be dense-in-itself topological spaces. If there is a continuous semi-open mapping f from X onto Y, then $n(X) \leq n(Y)$.*

Proof: If $\bigcup\mathcal{D} = Y$, then $\bigcup\{f^{-1}[D]\colon D \in \mathcal{D}\} = X$ for any onto mapping and any family $\mathcal{D}$ of subsets of Y. So it suffices to show that the preimage of a nowhere dense subset $D \subseteq Y$ is nowhere dense in X. We may assume that D is closed; then by the continuity, $f^{-1}[D]$ is closed, too. If $U = \mathrm{Int} f^{-1}[D] \neq \emptyset$, then $\emptyset \neq \mathrm{Int} f[U] \subseteq D$, because f is semi-open. But this contradicts the assumption that D is nowhere dense. ∎

COROLLARY:

(a) *If X is compact dense-in-itself, then $n(X) = n(E(X))$, where $E(X)$ denotes the Gleason space (absolute, projective cover) of X.*

(b) *If κ is a singular cardinal, then $n(U(\kappa)) \leq n(U(\mathrm{cf}(\kappa)))$.*

Proof: (a) The canonical projection $\pi\colon E(X) \to X$ is continuous, semi-open and onto. Thus $n(X) \geq n(E(X))$ by Lemma 2. However, π is also irreducible, which implies that the image of a closed nowhere dense subset of $E(X)$ is nowhere dense in X. Hence $n(X) \leq n(E(X))$

(b) Denote $\lambda = \mathrm{cf}(\kappa)$ and choose an onto mapping $f\colon \kappa \to \lambda$ such that for every $\xi < \eta < \lambda, |f^{-1}(\xi)| < |f^{-1}(\eta)| < \kappa$. Let $g = \beta f \restriction U(\kappa)$, where βf is the

Čech–Stone extension $\beta f\colon \beta\kappa \to \beta\lambda$. Then g maps $U(\kappa)$ continuously onto $U(\lambda)$. If $V \subseteq U(\kappa)$ is nonempty and open in $U(\kappa)$, then for some $M \subseteq \kappa$ with $|M| = \kappa$, $\mathrm{cl}_{\beta\kappa} M \cap U(\kappa) \subseteq V$. By our choice of the mapping f, clearly $|f[M]| = \lambda$, hence $g[V] \supseteq \mathrm{cl}_{\beta\lambda} f[M] \cap U(\lambda)$. The last set is nonempty and open in $U(\lambda)$, which shows that g is semi-open. Now, Lemma 2 applies. ∎

Now we are ready to present the first statement.

THEOREM 1: *If* $\mathrm{cf}(\kappa) > \omega$, *then* $n(U(\kappa)) = \omega_1$.

Proof: The inequality $n(U(\kappa)) \geq \omega_1$ follows by the Observation. If κ is a singular number, then $\omega_1 \leq n(U(\kappa)) \leq n(U(\mathrm{cf}(\kappa)))$ by the previous corollary. Therefore it suffices to show that $n(U(\kappa)) \leq \omega_1$ for κ regular uncountable.

Let us order the family ${}^\kappa\kappa$ by $f \leq g$ iff $|\{\xi < \kappa\colon f(\xi) > g(\xi)\}| < \kappa$ and put $\mathfrak{b}_\kappa = \min\{\{|F|\colon F \subset {}^\kappa\kappa$ has no upper bound$\}$. A standard diagonal argument shows that $\mathfrak{b}_\kappa \geq \kappa^+$.

The following statement together with full proof can be found in [BS, Theorem 2,8]:

Let κ be a regular uncountable cardinal. Then there is a collection $\{\mathcal{A}_{n,\alpha}\colon n \in \omega, \alpha \in \mathfrak{b}_\kappa\}$ such that:

(i) For each $n < \omega$, $\bigcup\{\mathcal{A}_{n,\alpha}\colon \alpha \in \mathfrak{b}_\kappa\}$ is a maximal UAD family on κ;

(ii) for each $n < \omega$, $\alpha < \beta < \mathfrak{b}_\kappa$, $\mathcal{A}_{n,\alpha} \cap \mathcal{A}_{n,\beta} = \emptyset$;

(iii) for every $M \in [\kappa]^\kappa$, there is some $n < \omega$ such that for each $\alpha < \mathfrak{b}_\kappa$, $|M \cap A| = \kappa$ for some $A \in \mathcal{A}_{n,\alpha}$.

Evidently, this is sufficient for the application of Lemma 1 with $\tau = \omega$. If we define $\mathcal{V}_n = \{V(n,\alpha)\colon \alpha < \mathfrak{b}_\kappa\}$, where $V(n,\alpha) = \bigcup\{\widehat{A}\colon A \in \mathcal{A}_{n,\alpha}\}$, then the family $\{\mathcal{V}_n\colon n < \omega\}$ satisfies all the assumptions of Lemma 1 — (i) and (ii) imply (a), (c) follows from (iii), and as $\mathfrak{b}_\kappa > \kappa \geq \omega^+$, (b) is satisfied, too. Thus $n(U(\kappa)) \leq \omega_1$. ∎

THEOREM 2: *If* $\kappa > \mathrm{cf}(\kappa) = \omega$, *then* $n(U(\kappa)) = \omega_2$, *provided that either*

(a) $\neg CH$, *or*

(b) $2^{\omega_1} = \omega_2$, *or*

(c) $\kappa^\omega = 2^\kappa$

holds.

Proof: Our aim is to find a family $\{\mathcal{V}_\alpha\colon \alpha < \omega_1\}$ satisfying the assumptions of Lemma 1. To do so, we shall slightly modify the proof of [BS, Theorem 2.7 (ii)]. To be precise, we shall construct a collection $\{\mathcal{A}_{\alpha,\xi}\colon \alpha < \omega_1, \xi < \mathfrak{c}\}$ such that

(i) for each $\alpha < \omega_1, \mathcal{A}_\alpha = \bigcup_{\xi<\mathfrak{c}} \mathcal{A}_{\alpha,\xi}$ is a maximal UAD family on κ,

(ii) for each $\alpha < \omega_1, \xi < \eta < \mathfrak{c}, \mathcal{A}_{\alpha,\xi} \cap \mathcal{A}_{\alpha,\eta} = \emptyset$,

(iii) for every $M \in [\kappa]^\kappa$ there exists an $\alpha < \omega_1$ such that for every $\xi < \mathfrak{c}$ there is some $A \in \mathcal{A}_{\alpha,\xi}$ with $|M \cap A| = \kappa$.

An analogous reasoning as in the proof of the previous theorem shows that Lemma 1 applies, *provided* $\mathfrak{c} > \omega_1$, since we can ensure only that $|\mathcal{V}_\alpha| = \mathfrak{c}$.

Assuming (c), we can construct a similar family $\{\mathcal{A}_{\alpha,\xi}: \alpha < \omega_1, \xi < 2^\kappa\}$ having the respective three properties, with 2^κ replacing $\mathfrak{c}$ everywhere. We conclude the proof showing the statement under (b).

The construction is done using transfinite induction. Simultaneously with $\{\mathcal{A}_\alpha: \alpha < \omega_1\}$ we shall find a family $\{f_A: A \in \mathcal{A}_a, \alpha < \omega_1\}$ where $f_A: A \to \kappa$ is a one-to-one function.

Let $\mathcal{A}_0 = \{\kappa\}$, let f_κ be the identity function on κ, let $\mathcal{A}_{0,0} = \mathcal{A}_0$, $\mathcal{A}_{0,\xi} = \emptyset$ for $0 < \xi < \mathfrak{c}$.

Let $\alpha < \omega_1$ and suppose that $\mathcal{A}_\alpha$ and all $f_A, A \in \mathcal{A}_\alpha$, are known. We notice the following:

Observation: *Whenever f_A: $A \to \kappa$ is a one-to-one function ($A \in [\kappa]^\kappa$) and $B \in [A]^\kappa$ then there is a set $C \in [B]^\kappa$ and a one-to-one mapping f_C: $C \to \kappa$ such that $f_C(\gamma) < f_A(\gamma)$ for all $\gamma \in C$.*

Indeed, let h be a one-to-one increasing mapping from $f_A[B]$ onto κ. Define $C = \{\gamma \in B: h(f_A(\gamma))$ is a successor ordinal$\}$ and for $\gamma \in C$, let $f_C(\gamma) = \delta$, where δ is the (unique) ordinal satisfying $h(\delta) + 1 = h(f_A(\gamma))$. This works.

By the observation, for each $A \in \mathcal{A}_\alpha$ one can find a maximal UAD family $\mathcal{A}(A)$ on A such that every $B \in \mathcal{A}(A)$ is a domain of a one-to-one function f_B satisfying $f_B(\gamma) < f_A(\gamma)$ for all $\gamma \in B$. Let $\mathcal{A}_{\alpha+1} = \bigcup\{\mathcal{A}(A): A \in \mathcal{A}_\alpha\}$. Finally, let $\mathcal{A}_{\alpha+1,0} = \mathcal{A}_{\alpha+1}, \mathcal{A}_{\alpha+1,\xi} = \emptyset$ for $0 < \xi < \mathfrak{c}$.

Let $\alpha < \omega_1$ be a limit ordinal and suppose that all $\mathcal{A}_\beta$ ($\beta < \alpha$) and f_A ($A \in \mathcal{A}_\beta, \beta < \alpha$) have been found. The inductive assumptions are as follows: if $\beta_0 < \beta_1 < \alpha, A_0 \in \mathcal{A}_{\beta_0}, A_1 \in \mathcal{A}_{\beta_1}$, then either $|A_0 \cap A_1| < \kappa$, or $|A_1 \setminus A_0| < \kappa$ and in this case $|A_0 \setminus A_1| = \kappa$ and $f_{A_1}(\gamma) < f_{A_0}(\gamma)$ for all $\gamma \in A_0 \cap A_1$.

Fix a strictly increasing sequence of regular cardinals $\langle \kappa_n: n \in \omega \rangle$ converging to κ and a strictly increasing sequence $\langle \alpha_n: n \in \omega \rangle$ of ordinals converging to α. Call a **chain** a family $\mathcal{C} = \{A_n: n \in \omega\}$, where $A_n \in \mathcal{A}_{\alpha_n}$ and $|A_{n+1} \setminus A_n| < \kappa$ for all $n \in \omega$.

Fix a chain $\mathcal{C} = \{A_n: n \in \omega\}$. Before proceeding further, we shall need two statements. The first one is analogous to the corresponding observation from the successor step of the induction.

FACT 1: *Let $B \in [\kappa]^\kappa$ be an arbitrary set satisfying $|B \setminus A_n| < \kappa$ for all $n \in \omega$. Then there is a set $C \in [B]^\kappa$ and a one-to-one mapping $f_C: C \to \kappa$ such that for all $n \in \omega$ and for all $\gamma \in A_n \cap C$ we have $f_C(\gamma) < f_{A_n}(\gamma)$.*

For $\gamma \in B \cap A_0$, define $g(\gamma) = \min\{f_{A_n}(\gamma): n \in \omega, \gamma \in A_n\}$. Since all mappings f_{A_n} are one-to-one, every preimage $g^{-1}(\delta)$ is at most countable. Let $D \subset B$ be any set satisfying $|D| = \kappa$ and $|D \cap g^{-1}(\delta)| \leq 1$ for all $\delta < \kappa$. Then $g \restriction D$ is one-to-one and $g(\gamma) \leq f_{A_n}(\gamma)$ for all $n \in \omega, \gamma \in D \cap A_n$. Applying the previous observation to D and $g \restriction D$, we can find $C \in [D]^\kappa$ and a one-to-one $f_C: C \to \kappa$ as required.

The second statement is Theorem A from [BV]. The reader is advised to consult the proof there. Let $\mathbf{Big} = \{Z \subseteq \omega\times\omega: \big|\{n \in \omega: \big|\{i \in \omega: (n,i) \in Z\}\big| = \omega\}\big| = \omega\}$.

FACT 2: *There is an almost disjoint family $\mathcal{T}$ consisting of graphs of infinite partial functions from ω to ω such that for every $Z \in \mathbf{Big}$, $|\{T \in \mathcal{T}: T \subseteq Z\}| = \mathfrak{c}$.*

Since $|\mathbf{Big}| \leq \mathfrak{c}$, a standard argument enables one to find a partition $\mathcal{T} = \bigcup_{\xi<\mathfrak{c}} \mathcal{T}_\xi$ such that for every $Z \in \mathbf{Big}$ and every $\xi < \mathfrak{c}$ there is some $T \in \mathcal{T}_\xi$ with $T \subseteq Z$. Let us fix such a partition and we may continue with the proof.

For $n \in \omega$, choose a partition $\{R_{n,i}: i \in \omega\}$ of the set $\bigcap_{j\leq n} A_j \setminus A_{n+1}$ such that $|R_{n,i}| = \kappa_i$. For $T \in \mathcal{T}$, let $B(T) = \bigcup\{R_{n,i}: (n,i) \in T\}$. According to our definitions, for every $T \in \mathcal{T}$ and for every $n \in \omega$, $|B(T) \setminus A_n| < \kappa$. Applying Fact 1, we can find a family $\mathcal{A}(\mathcal{C})$ with the properties as follows:

(1) The family $\mathcal{A}(\mathcal{C})$ is UAD,
(2) for every $A \in \mathcal{A}(\mathcal{C})$ and every $n \in \omega$, $|A \setminus A_n| < \kappa_n$,
(3) the family $\mathcal{A}(\mathcal{C})$ is the maximal one satisfying (1) and (2),
(4) for every $A \in \mathcal{A}(\mathcal{C})$ and every $T \in \mathcal{T}$, either $|A \setminus B(T)| < \kappa$ or $|A \cap B(T)| < \kappa$,
(5) for every $A \in \mathcal{A}(\mathcal{C})$ there is a one-to-one function $f_A: A \to \kappa$ such that for every $n \in \omega$ and every $\gamma \in A \cap A_n$, $f_A(\gamma) < f_{A_n}(\gamma)$.

Decompose the family $\mathcal{A}(\mathcal{C})$ as follows: For $0 < \xi < \mathfrak{c}$, let $\mathcal{A}(\mathcal{C},\xi) = \{A \in \mathcal{A}(\mathcal{C}): |A \setminus B(T)| < \kappa$ for some $T \in \mathcal{T}_\xi\}$, $\mathcal{A}(\mathcal{C},0) = \mathcal{A}(\mathcal{C}) \setminus \bigcup_{0<\xi<\mathfrak{c}} \mathcal{A}(\mathcal{C},\xi)$. Now it remains to define

$$\mathcal{A}_\alpha = \bigcup\{\mathcal{A}(\mathcal{C}): \mathcal{C} \text{ is a chain}\}, \quad \mathcal{A}_{\alpha,\xi} = \bigcup\{\mathcal{A}(\mathcal{C},\xi): \mathcal{C} \text{ is a chain}\} \quad \text{for } \xi < \mathfrak{c}.$$

The verification that $\mathcal{A}_\alpha$ is a maximal UAD family is easy and straightforward. This completes the inductive definitions.

We have to verify that (iii) holds, since (i) and (ii) are clearly true. Let $M \in [\kappa]^\kappa$ be arbitrary. First we shall show that there is some $\alpha < \omega_1$ such that $|M \cap A| = \kappa = |M \cap A'|$ for two distinct $A, A' \in \mathcal{A}_\alpha$.

Suppose the contrary. Since every $\mathcal{A}_\alpha$ is maximal, there is some $A_\alpha \in \mathcal{A}_\alpha$ such that $|A_\alpha \cap M| = \kappa$; by our assumption this A_α is unique. So $|M \setminus A_\alpha| < \kappa$. Since the cofinality of κ is countable, there is a cardinal $\tau < \kappa$ such that the set $J = \{\alpha < \omega_1 \colon |M \setminus A_\alpha| < \tau\}$ is uncountable. Since $\kappa > \tau \cdot \omega_1$, $|M \setminus \bigcap_{\alpha \in J} A_\alpha| < \kappa$. Hence the set $K = M \cap \bigcap_{\alpha \in J} A_\alpha$ is of full size κ. Choose a strictly increasing sequence $\langle \alpha_n \rangle$ ranging in J and $\gamma \in K$. Denoting $f_n = f_{A_{\alpha_n}}$, we get from the inductive construction a decreasing sequence of originals $f_0(\gamma) > f_1(\gamma) > f_2(\gamma) > \cdots$, a contradiction.

In order to show (iii), we start with finding a suitable $\alpha < \omega_1$. Let $\beta_0 < \omega_1$ be such that for some $B_0 \in \mathcal{A}_{\beta_0}$ one has $|M \cap B_0| = \kappa = |M \setminus B_0|$. Next, consider the set $M \cap B_0$ and find β_1 and $B_1 \in \mathcal{A}_{\beta_1}$ with $|M \cap B_0 \cap B_1| = \kappa = |M \cap B_0 \setminus B_1|$. Proceeding further, we shall find for all $n \in \omega$ an ordinal β_n and a set $B_n \in \mathcal{A}_{\beta_n}$ satisfying $|M \cap \bigcap_{i \le n} B_i| = \kappa = |M \cap \bigcap_{i<n} B_i \setminus B_n|$.

Let $\alpha = \sup_{n<\omega} \beta_n$. When the α-th induction step took place, we used a sequence $\langle \alpha_n \colon n \in \omega \rangle$ cofinal in α and we had to consider all possible chains. In particular, we dealt with the chain $\mathcal{C} = \langle A_n \rangle$, $A_n \in \mathcal{A}_{\alpha_n}$ having the property that $|A_n \cap B_m| = \kappa$ for all $n, m \in \omega$. For this $\mathcal{C}$ we clearly have $|M \cap A_n \setminus A_{n+1}| = \kappa$ for infinitely many $n \in \omega$. Therefore the set

$$Z = \{(n, i) \in \omega \times \omega \colon n < i \text{ and } |M \cap R_{n,i}| > \kappa_n\}$$

belongs to **Big**.

Let $\xi < \mathfrak{c}$ be arbitrary; using Fact 2 choose a set $T \in \mathcal{T}_\xi$ with $T \subseteq Z$ and consider the set $B(T)$. By the definition of the set Z, for every $(n, i) \in T$ one has $|M \cap R_{n,i}| > \kappa_n$. Since T is an infinite graph of a partial function, $|M \cap B(T)| = \kappa$. Since $\mathcal{A}(\mathcal{C})$ is maximal there is some $A \in \mathcal{A}(\mathcal{C})$ such that $|A \cap M \cap B(T)| = \kappa$. For this A, $|A \setminus B(T)| < \kappa$ by (4), and, as $T \in \mathcal{T}_\xi$, $A \in \mathcal{A}(\mathcal{C}, \xi)$. So we succeeded to find an $\alpha < \omega_1$ and $A \in \mathcal{A}_{\alpha,\xi}$ with $|A \cap M| = \kappa$.

Assume now that (c) holds, i.e., $\kappa^\omega = 2^\kappa$. Consider the family $\{\mathcal{A}_\alpha \colon \alpha < \omega_1\}$ constructed in the previous part of the proof. It was proved in [BS, 5.2–5.5] that once (1), (2), (3) and (5) are satisfied at every limit stage of the construction, then the following is true.

For every $M \in [\kappa]^\kappa$ *there is some* $\alpha < \omega_1$ *with* $|\{A \in \mathcal{A}_\alpha\colon |A \cap M| = \kappa\}| \geq \kappa^\omega$.

For the sake of completeness, we shall briefly repeat the proof from [BS].

We have already noticed that if $M \in [\kappa]^\kappa$, then for some $\alpha = \sup_{n\in\omega} \alpha_n < \omega_1$ and for a chain $\mathcal{C} = \{A_n\colon n \in \omega\}$ with $A_n \in \mathcal{A}_{\alpha_n}, |M \cap A_n \setminus A_{n+1}| = \kappa$ for infinitely many n's.

Now, since one has $|A \setminus A_n| < \kappa_n$ for every $A \in \mathcal{A}(\mathcal{C})$ and every $n \in \omega$ by (2), a standard diagonalization argument yields that $|\{A \in \mathcal{A}_\alpha\colon |M \cap A| = \kappa\}| \geq \kappa^+$. In order to pass from κ^+ to κ^ω, let us consider the minimal cardinal τ satisfying $\tau^\omega \geq \kappa^\omega$. Two cases are possible: Either $\tau = 2$ and then we are allowed to use (a); or $\mathrm{cf}\tau = \omega$. If the second possibility occurs, choose a strictly increasing sequence $\{\tau_n\colon n \in \omega\}$ of cardinals converging to τ.

The canonical tree of all functions φ with $\mathrm{dom}\,\varphi = n$ and $\varphi(i) \in \tau_i$ for $i < n\,(n \in \omega)$ has τ^ω branches. We shall define by induction two mappings F, G from all nodes of the tree. Choose $A \in \mathcal{A}_0$ with $|A \cap M| = \kappa$ and let $F(\emptyset) = A$, $G(\emptyset) = 0$. If $\mathrm{dom}\,\varphi = n$ and $F(\varphi) = A$ and $G(\varphi) = \alpha \in \omega_1$ are known and $A \in \mathcal{A}_\alpha, |A \cap M| = \kappa$ holds, then there is some β satisfying $\alpha < \beta < \omega_1$ and $|\{B \in \mathcal{A}_\beta\colon |B \cap A \cap M| = \kappa\}| \geq \kappa^+$. Define $G(\varphi \cup \langle n, \xi\rangle) = \beta$ for all $\xi \in \tau_n$. Then enumerate first τ_n sets from $\{B \in \mathcal{A}_\beta\colon |B \cap A \cap M| = \kappa\}$ as $\{A_\xi\colon \xi \in \tau_n\}$ and put $G(\varphi \cup \langle n, \xi\rangle) = A_\xi$.

The mapping G naturally induces a mapping from all branches in the tree to countable sequences in ω_1. The tree contains τ^ω branches and there are only ω_1^ω sequences in ω_1. By the minimality of τ and by $\tau > 2$ we have $\tau > \omega_1^\omega$. Thus there is an increasing sequence $\{\alpha_n\colon n \in \omega\}$ of countable ordinals such that the family of all branches $\emptyset \subset \varphi_0 \subset \cdots \varphi_n \subset \cdots$ satisfying $G(\varphi_n) = \alpha_n$ has the size at least τ^+. But as all τ_n's are strictly smaller than τ, there must be, in fact, τ^ω such branches. Since $\tau^\omega \geq \kappa^\omega$ and since F induces a mapping from branches of the tree to distinct chains in $\bigcup\{\mathcal{A}_\alpha\colon \alpha < \omega_1\}$, for $\alpha = \sup_{n\in\omega} \alpha_n$ we have that $|\{A \in \mathcal{A}_a\colon |A \cap M| = \kappa\}| \geq \kappa^\omega$.

Thus, under the assumption $\kappa^\omega = 2^\kappa$, all that we need is to split every $\mathcal{A}_\alpha$ into $\mathcal{A}_{\alpha,\xi}\,(\xi < 2^\kappa)$ in such a way that for every $M \in [\kappa]^\kappa$ there is some $\alpha < \omega_1$ such that for every $\xi < 2^k, |M \cap A| = \kappa$ for some $A \in \mathcal{A}_{\alpha,\xi}$.

This is done as follows. Given $\alpha < \omega_1$, denote by $\mathcal{M}$ the set of all $M \in [\kappa]^\kappa$ with $|\{A \in \mathcal{A}_a\colon |A \cap M| = \kappa\}| \geq \kappa^\omega$. Since $|\mathcal{M}| \leq 2^\kappa$ and since $\kappa^\omega = 2^\kappa$, a routine application of the disjoint refinement lemma [Ku] gives the desired partition.

To complete the proof, assume $2^{\omega_1} = \omega_2$. Then if $2^\omega = \omega_2$, then (a) applies. If $2^\omega = \omega_1$, then $\omega_2 = 2^{\mathfrak{c}} = |U(\omega)|$. The statement follows now by Lemma 2 and its Corollary, since every one-point subset of $U(\omega)$ is nowhere dense.

Concluding remarks

In the previous part of the paper we formulated some results concerning Boolean algebra $\mathcal{P}_\kappa(\kappa)$ ($= \mathcal{P}(\kappa)\,\mathrm{mod}[\kappa]^{<\kappa}$) in topological language. Primarily we are interested in $\mathcal{P}_\kappa(\kappa)$ as a forcing notion.

Recall that the Baire number of $\mathcal{P}_\kappa(\kappa)$ is the smallest size of a family $\mathcal{S}$ of partitions of unity with the following property:

> for any ultrafilter U on $\mathcal{P}_\kappa(\kappa)$ (this corresponds to uniform ultrafilters on $\mathcal{P}(\kappa)$) there exists a partition $R \in \mathcal{S}$ such that $U \cap R = \emptyset$.

For our calculation of Baire numbers of $\mathcal{P}_\kappa(\kappa)$'s for different κ's we used the result saying that these algebras as forcing notion collapse cardinal numbers. Let us summarize what is known of collapse of cardinals by $\mathcal{P}_\kappa(\kappa)$. $\lambda \to \mu$ denotes the fact that λ is collapsed to μ.

(i) For $\kappa = \omega$, $2^\omega \to \mathfrak{h}$ [BPS]

(ii) For κ uncountable and regular, $\mathfrak{b}_\kappa \to \omega$ [BS]

(iii) For κ singular with $\mathrm{cf}(\kappa) = \omega$, $2^\omega \to \omega_1$ [Theorem 2]

(iv) For κ singular with $\mathrm{cf}(\kappa) \neq \omega$, $\mathfrak{b}_{\mathrm{cf}(\kappa)} \to \omega$ [Theorem 1]

Under additional assumptions for singular cardinals more is known:

(v) For κ singular with $\mathrm{cf}(\kappa) = \omega$ and $\kappa^\omega = 2^\kappa$, $\kappa^\omega \to \omega_1$ [BS]

(vi) For κ singular with $\mathrm{cf}(\kappa) \neq \omega$ and $2^\kappa = \kappa^+$, $2^\kappa \to \omega$ [BS]

Let us finish with a reasonable conjecture in ZFC: for a singular cardinal κ with countable cofinality, $\kappa^\omega \to \omega_1$, and for a singular cardinal κ with an uncountable cofinality, $\kappa^+ \to \omega$.

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